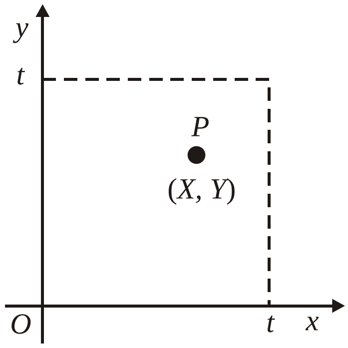
**1.**




The figure above shows a square of side $t$ and area $t^2$ which lies in the first quadrant with one vertex at the origin. A point $P$ with coordinates $(X, Y)$ is selected at random inside the square and the coordinates are used to estimate $t^2$. It is assumed that $X$ and $Y$ are independent random variables each having a continuous uniform distribution over the interval $[0, t]$.

[You may assume that $E(X^nY^n) = E(X^n)E(Y^n)$, where $n$ is a positive integer]

(a) Use integration to show that $E(X^n) = \dfrac{t^n}{n+1}$.

**(3)**

The random variable $S = kXY$, where $k$ is a constant, is an unbiased estimator for $t^2$.

(b) Find the value of $k$.

**(3)**

(c) Show that $\text{Var}(S) = \dfrac{7t^4}{9}$.

**(3)**

The random variable $U = q(X^2 + Y^2)$, where $q$ is a constant, is also an unbiased estimator for $t^2$.

(d) Show that the value of $q = \dfrac{3}{2}$.

**(3)**

(e) Find Var($U$).

**(3)**

(f) State, giving a reason, which of $S$ and $U$ is the better estimator of $t^2$.

**(1)**

The point (2, 3) is selected from inside the square.

(g) Use the estimator chosen in part (f) to find an estimate for the area of the square.

**(1)**
**(Total 17 marks)**

**1.** (a) $\int x^n \frac{1}{t}$ M1

$\int_0^t dx$ M1

$\mathrm{E}(x^n) = \int_0^t x^n \frac{1}{t} dx = \left[ \frac{x^{n+1}}{t(n+1)} \right]_0^t = \left( \frac{t^{n+1}}{t(n+1)} - 0 \right) \frac{t^n}{n+1}$ A1c.s.o. 3

(b) $\left( \mathrm{E}(x) = \frac{t}{2} \right)$ $\underline{\mathrm{E}(s) = k\mathrm{E}(x)\mathrm{E}(y),}$ $k.\frac{t^2}{4}$ <u>M1</u>, A1

$\mathrm{E}(s) = t^2$ $\Rightarrow k = 4$ A1 3

(c) $\mathrm{Var}(xy) = \mathrm{E}(x^2)\mathrm{E}(y^2) - [\mathrm{E}(xy)]^2$ M1

$= \frac{t^2}{3} \times \frac{t^2}{3} - \left( \frac{t^2}{4} \right)^2 = \left\{ \frac{7t^4}{144} \right\}$ M1

$\mathrm{Var}(s) = k^2 \mathrm{var}(xy) = 16 \times \frac{7t^4}{144} = \frac{7t^4}{9}$ A1 c.s.o. 3

(d) $\mathrm{E}(u) = t^2 \quad \Rightarrow 2\mathrm{E}(x^2)q = t^2, \quad \Rightarrow \frac{2t^2}{8} q = t^2, \quad \Rightarrow q = \frac{3}{2}$

M1, M1, A1 c.s.o. 3

| | | | |
|---|---|---|---|
| (e) | $\text{Var}(u) = q^2\left[\text{var}(x^2)+\text{var}(y^2)\right] = 2q^2\text{var}(x^2)$ | M1 | |
| | $\text{Var}(x^2) = \text{E}(x4) - \left[\text{E}(x^2)\right]^2 = \frac{t^4}{5} - \left(\frac{t^2}{3}\right)^2 = \left(\frac{4}{45}t^4\right)$ | M1 | |
| | $\text{Var}(u) = 2 \times \frac{9}{4} \times \frac{4}{45}t^4 = \frac{2}{5}t^4$ | A1 | 3 |
| (f) | $\frac{2}{5} < \frac{7}{9}$ $\therefore$ $u$ is better $\because$ smaller variance | B1ft | 1 |
| (g) | Using u estimate is $\frac{3}{2}\left(2^2+3^2\right) = \frac{3}{2} \times 13 = \frac{39}{2}$ or 19.5 | B1ft | 1 |

**[17]**

**1.** The structure and given answers helped many candidates here and those who attempted it were often able to pick up marks in at least parts (b), (d) and (g). Perhaps surprisingly part (a) proved to be the most challenging. There were many unconvincing attempts based on integrating $x^n$ and then for some reason dividing by $x$ but those who simply applied their S2 knowledge and wrote $\mathrm{E}(X^n) = \int_0^t x^n \frac{1}{t} dt$ were usually able to complete this part and often most of the question. A common error in part (c) was to assume Var($XY$)=Var($X$)Var($Y$) and this was perhaps the most challenging part. In part (f) the reasoning was usually sound, although the values were sometimes incorrect, and candidates who persevered to the end were often able to use the point (2, 3) in their estimator to answer part (g).